holosphere™

Special Edition: A Worksheet for the Benton Math

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A Holographer's Worksheet for the Benton Math*

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Introductory Remarks

Holographers recognize that the Benton math offers a powerful tool which can be used to control the image position, the viewing angle and the color of white light transmission (WLT) holograms. Unfortunately many holographers are not comfortable with the algebraic manipulation and trigonometric functions necessary to use the mathematical optics of WLT holograms. In an attempt to bridge some of these difficulties, an attached worksheet has been developed that presents the necessary formulas in the form and sequence that they are used. Utilizing this worksheet and a hand-held calculator**, specified parameters are substituted into the appropriate formulas, generating physical values for the design of a specific holocamera. Because this method takes into account both display and equipment constraints which will impact the final image, it should be of particular interest to holographers with limited equipment resources.

While this approach requires that holographers previsualize their work, it should not be viewed as rigid formula-generated imagery. Instead this method allows holographers greater flexibility in responding to holographic serendipity. It is often possible to modify the transfer camera and/or the display configuration to accomodate last minute changes, without having to redo a master. In doing so, holographers can gain a better understanding of the trade-offs involved when making changes. This knowledge will help them to design cameras that will afford them even greater flexibility. In addition, this method offers the technical precision that multicolor images and large-scale composite holograms require.

Before using the worksheet, it would be beneficial to review white light transmission holography and the function of the formulas used here. A hologram has the ability to diffract and focus light. Both of these characteristics are a function of the spacing and orientation of the fringe planes formed throughout the volume of the recording medium. When illuminated, a WLT hologram will reconstruct both an image of the original object and a real image of the slit aperture. The positions of these images are critical in defining the viewing characteristics of the hologram. The diffracting properties of the hologram will determine the viewing direction (θV) and the dominant color (λV) of the reconstructed image. The image of the object and the slit-image will form at a distance from the hologram determined by the focusing properties of the hologram. The diffraction formula used is discussed in the section on Grating vs. Bragg Models. The focusing properties are discussed in the section on the Hololens Formulas. The input parameters used in the worksheet are described in the last section. It is hoped that after reading this paper and looking over its accompanying diagrams, holographers will find the camera-design worksheet relatively easy to use. The concepts discussed in this paper presuppose a basic understanding of the techniques and theory of holography.

White Light Transmission Holography

Transmission holograms are usually not viewable in white light. This is because every finite area of the hologram contains the encoded information necessary to reconstruct an entire image from that viewpoint. When illuminated with white light. image points are focused at different distances for each illuminating wavelength (see fig. 1). Because the diffracting properties of a hologram are also wavelength dependent, a continuum of different colored images from slightly different viewpoints are superimposed. The result is a spectrally smeared image. The extent of this image smearing is in part related to the distance between the object points and the hologram. As the distance increases, so does the image smearing. By recording the interference pattern formed by a real image of the object focused in the plane of the hologram, these distances and the resulting smearing can be kept to a minimum. If the object depth does not exceed several centimeters, the small amount of accompanying spectral smearing is not detrimental to the image reconstruction and full parallax can be enjoyed. When a lens is used to form the real image, this type of white light hologram is called a focused-image hologram. If the real image is formed holographically, then the hologram is termed an open-aperture WLT.

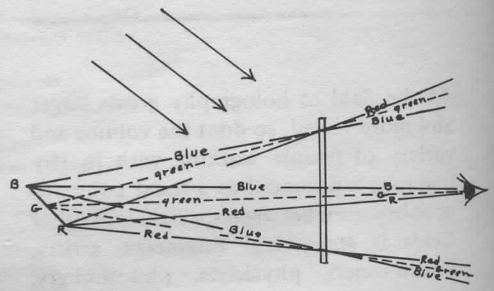

Fig. 1: Information about an object point is encoded throughout the hologram in a laser-viewable transmission hologram. If this hologram is illuminated with white light, each wavelength forms its own set of image points. The perceived result is that each image point is smeared into a spectral ellipse.

The Benton technique overcomes the depth limitation of the focused-image hologram.(1) In most recording situations, information about out-of plane points are encoded over a greater area of the recording medium than in-plane points. The greater the point's distance, the greater the area (see fig.2). By introducing a horizontal slit aperture, object information for each point is confined to a thin horizontal strip. This strip contains full left-to-right parallax informa-

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* Benton, S.A., *The Mathematical Optics of White Light Transmission Holograms*, proceedings of first International Symposium on Display Holography. Lake Forest College, 1982. Ed.: Dr. T. Jeong.

** An appropriate calculator will have scientific functions such as $1/x$, x^2, square root, normal and inverse trig functions (i.e. $\sin^{-1}$). An inverse trig function returns the angle when the value of the trig function is known. It is also useful but not essential to have several memory registers. Calculators with these functions can be purchased from about $20.

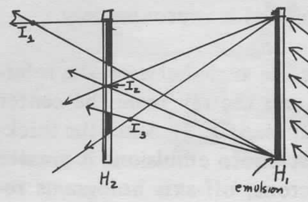


Fig. 2a: Distribution of image information in H^2 in open-aperture transfer image point I^2 (focused at H^2) is encoded in a very small area of the recording medium; points I^1 and I^3 that are farther from the hologram distribute the encoded image information over a greater portion of H^2.

tion, but has very little vertical information. Because of this restriction, the vertical distribution of encoded information corresponds to different points of the object rather than different viewpoints of the entire object and becomes more like a photograph top to bottom. Eliminating vertical parallax reduces the coherence requirements of the illuminating source (degree of monochromaticity). This makes it possible to produce holographic images with substantial depth, that are viewable in white light.

The procedure for making a Benton hologram requires two steps. A transmission master (called H^1 in this paper) is recorded and processed. The H^1 is illuminated with the conjugate of the original reference beam, producing a real image of the object. A masking slit is placed in front of H^1 to restrict the image information in the vertical direction. A second holographic plate, H^2, is placed so that the real image of H^1 straddles H^2. The object wavefront interferes with the reference beam for H^2, forming a hologram that, after processing, can be viewed with white light. When H^2 is displayed, it will be back-illuminated by a beam that is the conjugate of the reference beam used to record the WLT. This will reconstruct an image of the object straddling the plate and a real image of the slit (see fig. 3).

Conjugate Light Sources

In order to successfully record a WLT it is necessary to understand the concept of conjugate beams and the use of their resulting conjugate images. If an exposed and developed volume transmission hologram is placed in its original re-

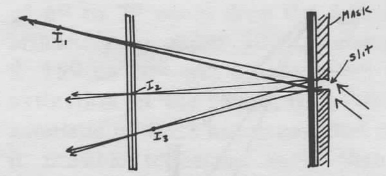


Fig. 2b: A horizontal slit placed in front of the master, H^1, reduces the amount of vertical information recorded in H^2.

cording position and illuminated by the same reference beam, an aberration-free virtual image of the object will be formed at the object's original position (unlike thin holograms, thick – volume – holograms will form only one reconstructed image for each illuminating direction. Thin holograms reconstruct both a primary and a conjugate image for each illuminating direction). See fig. 4. If the illuminating beam could be time-reversed (i.e., travel exactly the same path, but in the opposite direction), the resulting image wavefront would also be reversed. Instead of producing the diverging wavefront of a virtual image, this conjugate illumination would produce a converging object wavefront, such as that formed by a lens. Therefore at the location of the original object a real, conjugate image of that object would be formed. This image could either be seen on a screen, or viewed from a position beyond the image focal space where it becomes a diverging wavefront that our eyes can focus. Because of the lens-like nature of the hologram, the depth of the image would appear reversed or pseudoscopic.

Lenses can be used to produce conjugate beams in the following manner. If a point source of light is located at the focal point of a lens, the light passing through the lens will emerge as parallel rays (see fig. 5a). This collimated light can be thought of as a plane wavefront formed by a light source which is an infinite distance away. If a collimated beam is used as a reference source, a conjugate illuminating beam can be produced by turning the hologram upside-down so that it is illuminated from the back (see fig. 5b). Unfortunately the production of a wide beam of collimated

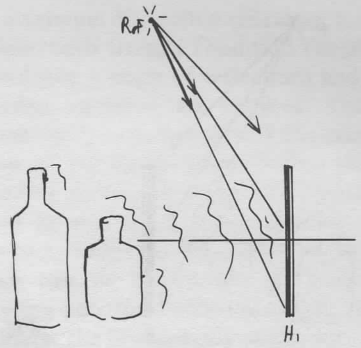


Fig. 3a: Recording the transmission master.

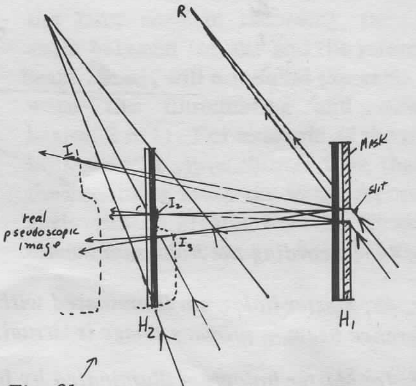


Fig. 3b

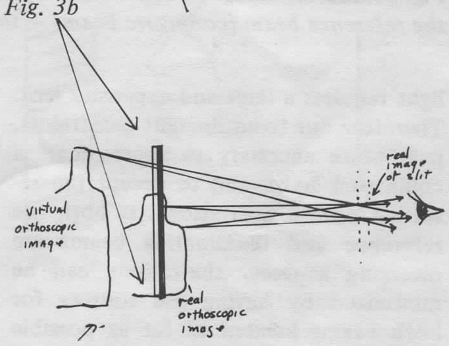


Fig. 3c

Fig. 3b: Illuminating the master, H^1, with a conjugate beam, producing a real image straddling H^2. A mask with a horizontal slit is placed in front of H^1 to reduce the object information. A reference beam for H^2 is introduced; this is the conjugate of the illuminating beam to be used for display.

Fig. 3c: The exposed and developed WLT, H^2, is illuminated, producing an image which straddles the film plane. A real image of the slit aperture is formed in front of the hologram.

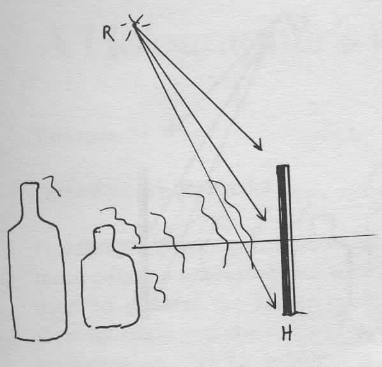
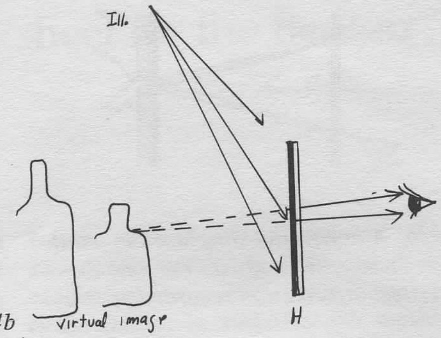


Fig. 4a

Fig. 4b



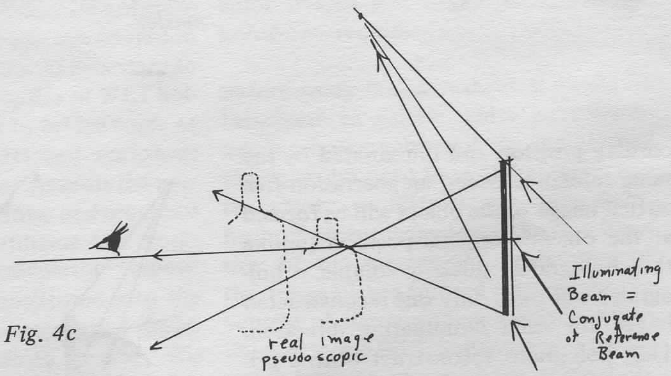


Fig. 4c

Fig. 4a: Recording the hologram master.

Fig. 4b: Master hologram illuminated with an illuminating beam identical to the reference beam – primary image is virtual.

Fig. 4c: Master hologram illuminated by illuminating beam that is time-reverse of the reference beam (conjugate beam) – the conjugate beam is a real image.

light requires a large and expensive lens. Therefore due to equipment constraints, it is often necessary to approximate a collimated beam and to accept the resulting optical aberrations. If both the reference and illuminating beams are diverging sources, aberrations can be minimized by having the sources for both beams located as far as possible from the hologram. Benton suggests that the beam distance should be at least ten times the length of the largest plate dimension.

The reference beam used for H^2 should be the conjugate of the illuminating beam that will be used for display. If the final WLT is to be illuminated by a diverging light source, then ideally the reference beam used for H^2 should be a converging beam (see fig. 6). Producing a converging beam, however, requires two large and expensive lenses. As in the previous case, the hologram can be back-illuminated and the aberrations that result from the differences between the illuminating beam and the conjugate of the reference beam can be minimized by making the reference and the illuminating distances as long as possible.

When H^2 is back-illuminated, a real image of the slit is projected in front of the hologram, where the pupil of the eye can intercept it. Because the illuminating beam is the conjugate of the reference beam, the reconstructed image of the object is a conjugate image. However, because the object recorded in H^2 was a real pseudoscopic image, the image reconstructed will be a virtual image with normal depth (see fig. 3).

Grating vs. Bragg Models

It is possible to calculate the distance (d) between interference fringe maxima using either one of two formulas:

1) The grating equation:

$$dg = \lambda/(\sin\theta_1 + \sin\theta_2)$$

2) Bragg's law: $db = \lambda/(2 \sin\beta)$ where $\beta = \frac{\theta_1 + \theta_2}{2}$. When the thickness of the recording medium is less than the average fringe spacing, the hologram is considered a thin hologram and the grating equation should be used. The average fringe spacing (d) is approximately

$$d \approx \lambda/\sin 2\beta$$

where 2β is the angle between the reference beam and the ray from the center of the object (see fig. 7). Since the thickness of most photo emulsions is greater than 3 microns, off-axis holograms recorded on photo emulsions are thick (or "volume") holograms.

Referring to fig. 7c, note that if ϕ is the angle that the fringes make with the normal to the hologram, then

$$\cos\phi = db/dg$$

This implies that dB=dG (and the two formulas are interchangeable) *only* when $\phi = 0°$ and the fringes are normal to the plane of the hologram (since dB will always be less than or equal to dG, using dG to determine the spatial frequency of the fringes will often underestimate the resolution needed in the recording medium). The fringes at any point on the hologram will lie on the bisector of the object/reference angle (line OB in fig.7d) therefore the fringes will be normal to the hologram when $\theta_1 = \theta_2$ (i.e. when the illuminating angle = the viewing angle or the reference angle = the

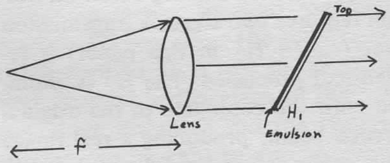


Fig. 5a: Recording with plane wave.

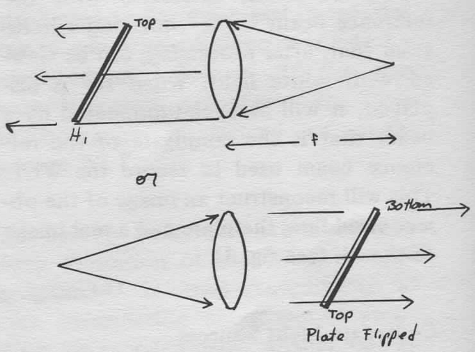


Fig. 5b: Illuminating with conjugate beam – by moving the light sources and lens to the other side of it or 2c – flipping H^1 upside-down.

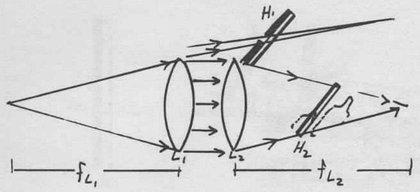


Fig. 6: Recording with a converging reference beam, which is the conjugate of the illuminating beam to be used.

object angle). This particular configuration also has the advantages of keeping the emulsion shrinkage and its accompanying distortions to a minimum. Benton states that it also minimizes distortions due to astigmatism.[2]

Even when the configuration will permit the use of the grating equation it is wise to keep in mind that you are working with volume holograms. There are significant differences in the characteristics of thin and thick holograms, particularly their behavior when illuminated by a light source that differs from that used in the recording process.

Diffraction Efficiency and the WLT

When a *thin* hologram is illuminated, the wavelength and/or illuminating angle can be made to differ substantially from that used in the recording process, with little loss in diffraction efficiency. In thin holograms, changes in the illuminating source will be compensated for by viewing angle changes. However the diffraction efficiency of thick holograms is sensitive to changes in wavelength and changes in the angle of the illuminating beam (the Bragg angle is the angle (β) that the light should make with the fringe planes for maximum diffraction efficiency). The greater the object/reference angle (2β), the greater this sensitivity. This produces substantial decreases in the diffraction efficiency when the illuminating beam differs from the reference beam (diffraction efficiency is a measure of how much of the illuminating light is diffracted into the image wavefront). Consider for example a typical holographic configuration where θI is between 40° and 50°, θV is equal to 0°, and the photo emulsion of 6-7 microns thick. A change in the illuminating angle of ± 3° to 4° could reduce the diffraction efficiency to about 80%. A change of 6° to 7° could drop the diffraction efficiency to about 50%. A change of ± 15° to 20° will usually result in the extinction of the image. It is this characteristic of thick holograms that makes it possible to record more than one image on a hologram, and to view each image separately.

A similar loss in diffraction efficiency will result from changes in wavelength between the recording and illuminating processes. If the object beam is a plane wave, it is possible to compensate for such changes in wavelength by deviating the direction of the illuminating beam. However, in the case of a complex object, deviations in the illuminating beam would benefit the reconstruction of some points on the object at the expense of other points on the object. When the wavelength changes, it is difficult to find a single illuminating beam that can satisfy the Bragg condition for every object point's resulting fringe pattern. In general, a complete image will not be formed if such changes in wavelength occur, unless some precautions are taken.[3]

A maximum diffraction efficiency results when both Bragg's condition (angle of incidence = angle of reflection) and the grating equation are fulfilled. This is most easily accomplished if the orientation of the fringe planes within the recording medium is taken into consideration as well as the fringe spacing. The correct fringe spacing needed at the surface can be determined by using the grating equation with the display parameters: the illuminating angle, θI , the viewing angle, θV , and the viewing wavelength, λV . The correct orientation for the fringe planes, ϕ , is

$$\phi = \frac{\theta I - \theta V}{2}$$

If the viewing color differs from that of the laser used in recording, then the angle between the slit and the reference beam, $2\beta rec$, will not equal the angle between the illuminating and viewing beams, $2\ \beta ill$. For example, if the playback wavelength is shorter than that of the recording laser, the Bragg recording angle will be greater than the Bragg illuminating angle, $\beta rec > \beta ill$ (see fig. 8).

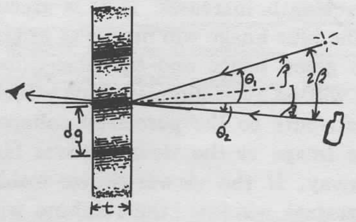


Fig. 7a

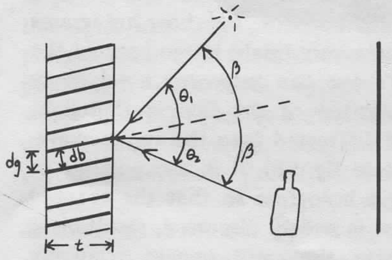


Fig. 7b

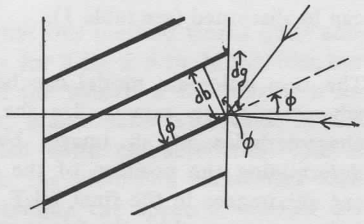


Fig. 7c

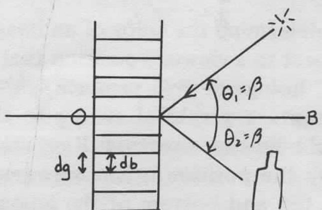


Fig. 7d

Fig. 7a: Thin hologram.

Fig. 7b: Volume, thick hologram.

Fig. 7c: The fringe spacing at the surface of the emulsion, dG, is always greater than or equal to the spacing between the fringe planes, dB.

Fig. 7d: Fringes perpendicular to the plane of the hologram.

To maintain the correct fringe orientation, the reference angle,

$\theta R = \beta rec + \phi$

and the slit angle,

$\theta S = \beta rec - \phi$

This insures that the fringe planes will bisect the illumination viewing angle, therefore meeting the Bragg condition. If the appropriate fringe spacing, dB, is recorded, the difference in the path-lengths traveled by individual rays will be m(λV), where m equals any integer, thus insuring that there will be constructive interference for the desired viewing color and direction. This provides the conditions for maximum diffraction efficiency and produces the brightest image. The entire image will be reconstructed as long as the spatial frequency of the fringes does not exceed the resolution of the film.

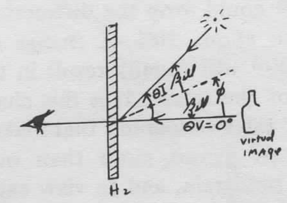


Fig. 8a: Illumination conditions desired

$$dg = \frac{\lambda V}{(\sin \theta I + \sin \theta V)}$$

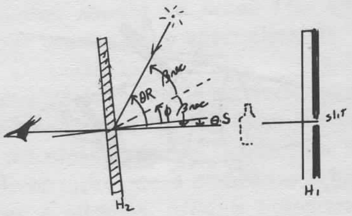


Fig. 8b: Recording the desired fringes where $\lambda V < \lambda L$ *and therefore* $\beta rec > \beta ill$.

The Spectral Window

When illuminated with white light, the image of the object will appear as if seen through a vertically dispersed spectral window, producing the familiar rainbow image (see fig. 9a). The spectral window is composed of the slit images formed by all the different wavelengths in the light source. Rainbow holograms produce a very bright image because the viewer's eye can intercept a relatively large portion of the illuminating light that is diffracted into the image wavefront (see fig. 9b). It is also possible to design a hologram so that the spectral window is widely dispersed, resulting in an image that will remain relatively monochromatic over a range of viewing distances, as in fig. 9 (note that when a monochromatic image is desired, the task is much easier if the object height is small with respect to the plate height).

To determine the color of an image with respect to a viewer's position that a planned hologram will produce, this paper presents a graphical analog to the real world viewing situation. Rays are drawn from the position of the viewer's eye to the top and bottom of the hologram (if the viewer's position is critical, wherein inaccuracies of several inches cannot be tolerated, then the diameter of the viewer's pupil cannot be ignored. In this case the rays are drawn from the bottom of the pupil to the top of the hologram and from the top of the pupil to the bottom of the hologram). Each ray crosses the spectral axis at a point corresponding to the slit image of its reconstructing wavelength, λ_i. As the viewer moves farther from the ideal viewing position (just behind the spectral axis), the image rays intercept a wider portion of the spectral window, and the image becomes less monochromatic and more rainbow-hued. When the rays intercept the spectral axis at wavelengths beyond the sensitivity range of our visual system, no image information is received. This corresponds to the apparent collapse of the image with increased viewing distnace. Note that because most holograms recorded on photo-emulsions are really volume holograms, the diffraction efficiency for colors other than the dominant viewing color can also decrease as the difference in wavelength increases. If V is green, then the blue image will not be as bright as the green image and the blue-violet image will be even dimmer. This will also contribute to the perceived collapse of the image as the viewer moves farther away. If the viewer moves inside the spectral window, the rainbow will reverse top to bottom as is indicated by the rays traveling to position no.1 in fig. 10. These rays are extended beyond the viewer until they intercept the spectral axis where the color information can be discerned (see table 1).

The thin hologram model can be used when trying to previsualize the color characteristics of an image, by first determining the position of the different slit images in the final WLT. If the hologram were thin, the fringe spacing, dG, at the surface of the hologram would be

$dg = \lambda/(\sin \theta_1 + \sin \theta_2)$

When a rainbow hologram is illuminated with white light, each wavelength reconstructs the entire image – but each of these reconstructed wavefronts must pass through their own slit image. The location of the array of slit images that make up the hologram's spectral window is most easily found by first determining α [alpha] , the angle that the axis of the spectral window makes with the normal, where

$\alpha = \tan^{-1}(\sin \theta I)$

One first draws the dominant viewing ray at an angle θV to the hologram's normal. The viewing distance, VD, is then measured from the center of the dominant color's slit image (G in fig. 11a). In the diagram the dominant wavelength is green, therefore all the green light diffracted by the hologram must pass through this green slit image. Next the axis of the spectral window is constructed so that it makes an angle, α , with the normal and intersects the dominant viewing ray at G (see fig. 11a).

One can then calibrate the spectral window by first finding the visually useful boundries (i.e., λ upper = .67μ ; lower = .42μ). Using

$$\theta V = \sin^{-1}\left(\frac{\lambda}{dg\ SH} - \sin \theta I\right)$$

where SH is a shrinkage factor that depends somewhat on recording geometry and greatly on processing. SH usually lies between .94 and .98 (the worksheet includes the shrinkage factor when determining the grating spacing to be recorded). In the above formula dG (SH) is equal to the grating spacing that exists after the hologram is processed. Next a convenient angular interval, for example every 5°, is chosen. Using the formula

$\lambda_i = dg\ SH\ (\sin \theta V_i + \sin \theta I)$

the wavelengths of the slit images that are diffracted through the set of angles chosen are then indicated on the axis (see fig. 11b).

To determine the image color from any position, rays are drawn from the viewer's eye to the top and bottom of the hologram and the color of the image is indicated by the wavelength of the point of intersection of the ray and the calibrated spectral axis (see Table 1).

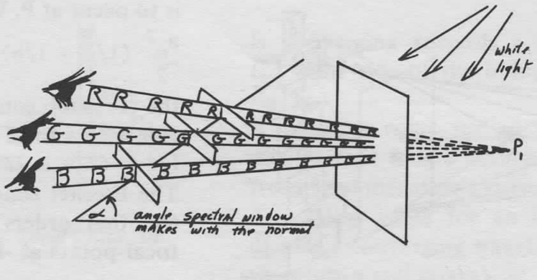


Fig. 9a: When a rainbow hologram is illuminated with white light, each wavelength reconstructs the entire image, but each of these reconstructed wavefronts must pass through their own slit image. Viewers of different heights will therefore see a reconstructed image point in different colors.

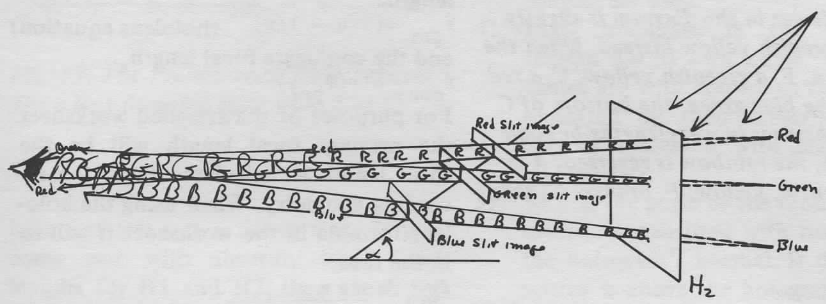


Fig. 9b: In a rainbow hologram the image remains monochromatic only when the viewer is relatively close to the spectral window. Otherwise the familiar rainbow-hued image is not seen.

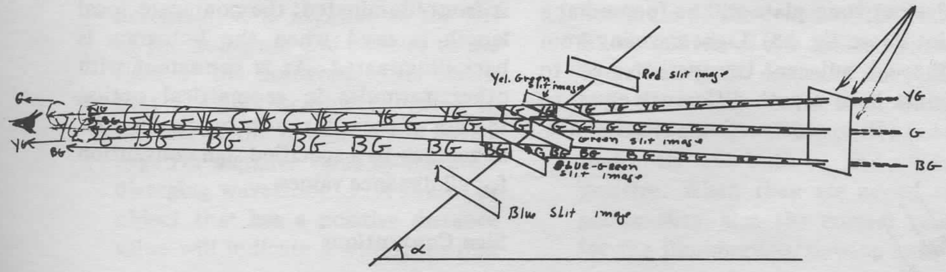


Fig. 9c: A more monochromatic image results when the illuminating angle is steeper and the slit is imaged farther from the hologram. The rays of light that reach the viewer's eye to form the image pass through a narrower portion of the spectral window.

Table 1

Wavelength in microns	*Perceived Color*
.380-.430	Bluish-purple
.430-.460	Purplish-blue
.460-.482	Blue
.482-.487	Greenish-blue
.487-.493	Cyan
.493-.497	Bluish-green
.497-.530	Green
.530-.558	Yellowish-green
.558-.570	Yellow-green
.570-.575	Greenish-yellow
.575-.580	Yellow
.580-.586	Yellow-orange
.586-.597	Orange
.597-.620	Reddish-orange
.620-.700	Red

Note that this method works quite adequately for 4 in. x 5 in. holograms, but that as the height of the hologram increases, inaccuracies will result in the predicted color characteristics. This is because the recording parameters used to determine the spectral window are only valid for information recorded near the center of the plate. Depending on the recording geometry these parameters and therefore the spectral windows for the top and bottom of the hologram may differ substantially. For example when the illuminating source is from above, the illuminating angle and distance is smaller at the top of the hologram and larger at the bottom. If the illuminating source is greater than five feet from the hologram, the change in the illuminating angle is usually less than $\pm$ 2.5° and can safely be ignored for most work. Similarly if the viewing distance is greater than three feet, then the change in the viewing angle is usually less than $\pm$ 4° and this simplified previsualization technique will usually prove adequate. However, if the illuminating and/or viewing distances are relatively short, or if the scale of the hologram is large, or if an image element covers the entire vertical extent of the hologram, it would be advisable to partition the hologram into different vertical segments and to find the spectral window for each segment.(4)

The Hololens Equations

A simple Gabor zone plate is formed when a recording medium is exposed to the interference pattern formed by two spherical wavefronts. In fig. 12, assume that S is the source of a reference wave and P is a point on the object that is illuminated by S and that then reflects light towards the hologram. Fringe maxima will occur in the recording medium at points Qm, each a distance, x_m from the optic axis (the center of the zone plate). These points, Qm, occur where pathlength differences between SQm and SPQm are equal to a whole number of wavelengths ($m\lambda$). Using the Pythagorean theorem the path from S to Qm is equal to $\sqrt{v^2 + x_m^2}$, and the distance from P to Qm is equal to $\sqrt{u^2 + x_m^2}$. Therefore the difference between the two path lengths is equal to

$$v - u + \sqrt{u^2 + x_m^2} - \left(\sqrt{v^2 + x_m^2}\right)$$

Because x_m is very small compared to both u and v,

$$\sqrt{u^2 + x_m^2} \simeq u + \left(x_m^2/2u\right)$$

and,

$$\sqrt{v^2 + x_m^2} \simeq v + \left(x_m^2/2v\right)$$

therefore the difference between the pathlengths is equal to

$$\frac{x_m^2}{2}\,(1/u - 1/v)$$

Since Qm is the set of points Q1, Q2, Q3, . . . where fringe maxima occur, then the path difference must equal a whole number of wavelengths, , and thus

$$m\lambda \simeq \frac{x_m^2}{2}\,(1/u - 1/v)$$

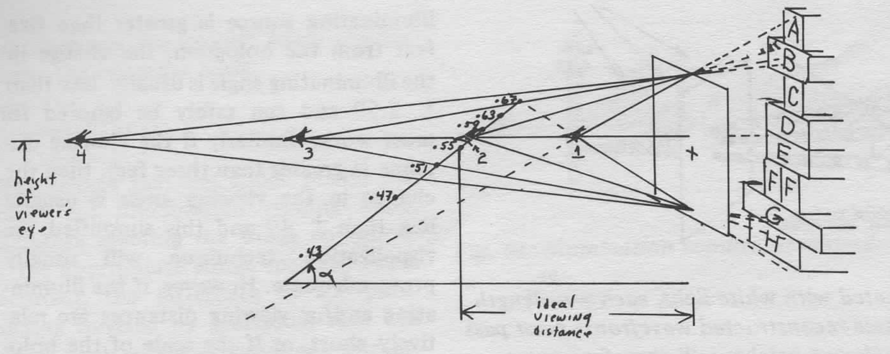


Fig. 10: This hologram was designed so that a viewer whose eye is at the same height as the center of the plate will see the entire image of the blocks in a yellowish green when viewed from the ideal viewing position. The viewer in the diagram is slightly taller, so at position no. 2 the image will appear a greenish yellow instead. When the viewer moves to position no. 3, block G will be green; E, a greenish yellow; C, a red; and B, a very dark red. From position no. 4, G will be blue-green; the bottom of C will be red and the top a dark red; while B will reconstruct in wavelengths beyond the visible and will not be seen. From position no. 1, the rainbow is reversed; A is a dim blue-violet; B, a blue; C, green; D, yellow-green; E, yellow; F, orange; G, red; and H, dark red.

If we let $(1/u - 1/v) = 1/f$, then fringes will be formed at $x_m = \sqrt{2m\lambda f}$. The first fringe will form where m = 1 and therefore the distance from the center of the zone plate, x_1, will equal $\sqrt{f\,2}$; the second fringe will form at a distance, $x_2 = \sqrt{f\,4}$; the third at $x_3 = \sqrt{f\,6}$; etc. The radii of the fringes is thus seen to be proportional to the square of the even integers. (5)

The above formulation is the same relationship required to form a Fresnel zone plate. A Fresnel zone plate has alternating transparent and opaque rings. Light traveling from a source, S, through a Fresnel zone plate will be focused at a point P (see fig. 13). Light traveling from S through adjacent transparent areas to P must have a path difference equal to $m\lambda$, if constructive interference is to occur at P. When

$$\frac{s_m^2}{2}(1/a + 1/b) = m\lambda$$

the Fresnel zone plate behaves like a lens with a focal length, f, such that

$$1/f = (1/a + 1/b)$$

The Fresnel zone plate diffracts light into other orders as well, so it also has focal points at -f, ± f/3, ± f/5, + f/7.(6)

Because the Gabor Zone plate has a sinusoidal fringe pattern rather than a square wave pattern, it has only two focal points ± f, where the primary focal length,

$$f_{pri} = (1/u - 1/v)^{-1} \text{ (hololens equation)}$$

and the conjugate focal length,

$$f_{conj} = -(f_{pri})$$

For purposes of the attached worksheet the primary focal length will be the focal length as calculated from the recording geometry. When using the hololens formula in the worksheet it will be in the form

$$f_{pri} = \frac{u \cdot v}{v-u}$$

When using the hololens formulas, the following sign convention should be used: the primary focal length is used during recording and when the hologram is front-illuminated; the conjugate focal length is used when the hologram is back-illuminated. As is consistent with other formulas in geometrical optics, the use of these formulas requires a *strict* adherence to a specified sign convention for all distance values.

Sign Conventions

The sign convention for this worksheet was chosen so that it would be consistent with that used in most articles published on related holographic topics. This will insure this worksheet's compatibility with formulas for evaluating image characteristics not covered in this paper. The main disadvantage of the sign convention used is that many of the distances will have negative values. While this may cause a little conceptual stumbling at first, the rules are relatively straightforward and cover all cases. For this reason the use of a sign convention that students would find more palatable was rejected. If you can hold onto the rule that diverging wavefronts are always negative and converging wavefronts are always positive, you should have very little difficulty using this system. When using the worksheet at first, however, you would do well to double-check all distance values to make sure that you

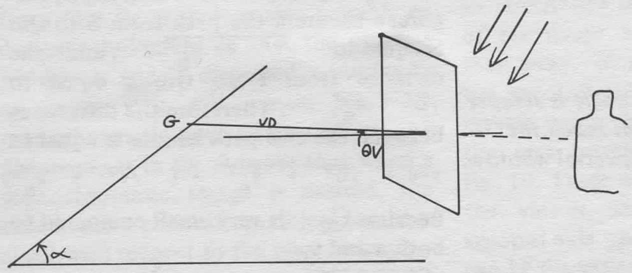


Fig. 11a: Constructing the axis of the spectral window using VD, ΘV *and* α.

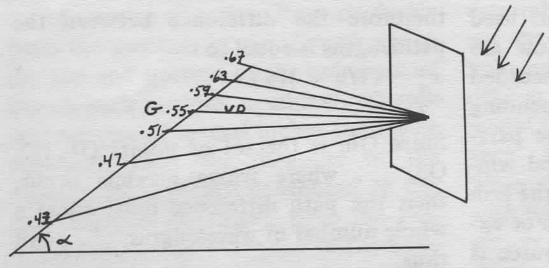


Fig. 11b: Calibrating the spectral axis in 6° increments.

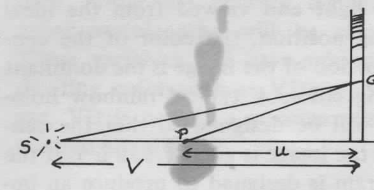


Fig. 12: Forming a Gabor zone plate; in the diagram m = 1.

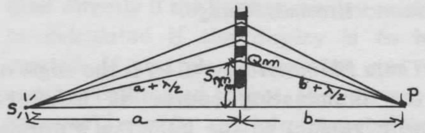


Fig. 13: The Fresnel zone plate behaves like a lens focusing light from S at P.

have not left out a minus sign. If you come out with absurdly large focal lengths for H1 and H2, then check for sign convention errors.

1) Light travels from left to right, as does the x-axis. The value for z (the distance) is zero at the plane of the hologram. Z is negative to the left of the hologram and positive to the right of the hologram. This means that a source or object located to the left of the hologram will have a negative distance value indicating a diverging wavefront . . . A source or object that has a positive distance value will indicate a wavefront that is converging towards a point on the right side of the hologram (see fig. 14).

2) A negative value for an image indicates that it is a diverging wavefront (and therefore a virtual image). A positive value for an image indicates a converging wavefront (and therefore a real image).

3) The distance value for collimated light is ∞ (infinite). Recall that $1/\infty = 0$; thus many calculations are simplified when collimated light sources are used.

4) Each hologram in the transfer camera has its own z-axis. Variable values are with respect to the z-axis indicated by their subscript (e.g., PD_1 is evaluated with respect to H_1's z-axis).

5) θ_I is the angle of the illuminating source as measured with respect to the hologram's normal. If the light source is above the hologram, then θ_I is positive; if the source is below, then θ_I is negative (see fig.15).

6) θ_V , the angle of view can be thought of as the angle between the line of sight and the normal to the hologram at the center of the plate. If the center of the image appears to be below the center of the plate, and the viewer must look downwards, then θ_V is positive (note that in fig. 15b both θ_I and θ_V are positive. When they are added together they give the correct value for the illumination/viewing angle). If the center of the image coincides with the center of the plate, then $\theta_V = 0$. If the center of the image appears to be above the center of the plate, then θ_V is negative.

7) ϕ , phi, the angle the fringes make with the normal, will usually have the same sign as θ_I . To determine whether ϕ is positive or negative, draw the perpendicular bisector of the illuminating/viewing angle, ill. If it lies above the normal, ϕ is positive, if it falls below the normal, ϕ is negative.

8) *If *both* θ_I and θ_V are negative (as in fig. 15a), *both* values can be made positive by reversing the criteria for 5, 6, and 7.

9) θ_R is governed by the same sign convention as θ_I .

10) θ_O is governed by the same sign convention as θ_V .

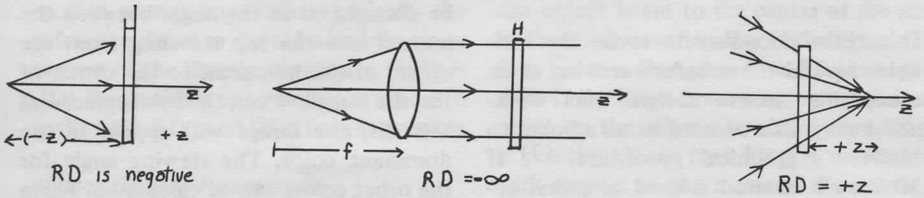


Fig. 14: Sign convention used in worksheet.

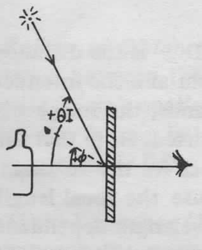


Fig. 15a: θ_I *is positive.*
$\theta_V = 0$.
ϕ *is positive.*

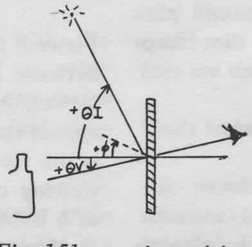


Fig. 15b: θ_I *is positive.*
θ_V *is positive.*
ϕ *is positive.*

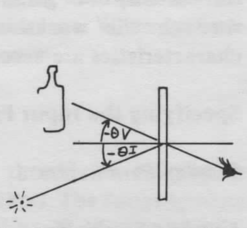


Fig. 15c: θ_I *is negative.*
θ_V *is negative.*
$\phi = 0$.

Using the Worksheet

The usual procedure when using the Benton math is to start from the desired illuminating and viewing geometry of the final hologram and to work backwards (in terms of how the hologram will be made). One first designs the transfer camera and then designs the master camera. This procedure is followed in the worksheet.

To use the worksheet, first enter the display parameters for the final WLT: the dominant viewing color, the illuminating angle, the viewing angle, the viewing distance and the illuminating distance. The first section (lines No. 1-6) is included to allow holographers to predetermine the color characteristics of the planned hologram. The values found can be used as outlined previously in the section, **Spectral Window**. Using this portion of the worksheet is optional.

Next, the wavelength of the recording laser, the shrinkage factor, and the reference distance for H_2 are entered where indicated. Using the formulas in steps No. 7-17, the angles and distances needed in the transfer camera are determined. Before choosing the master camera parameters, it is usually a good idea to sketch out the portion of the transfer camera as specified thus far. Table constraints could be a factor in determining the best position for H_1 when illuminated in the transfer camera. If the center of H_1 meets the end of the SD vector (see fig. 16), the master can be rotated to the most advantageous position with the following constraints in mind. When H_1 is illuminated, the zero order (that portion of the illuminating beam which passes through H_1 undiffracted) must not pass through H_2. As discussed previously, there are advantages in keeping $\phi = 0^o$ and in having the angle between the object and reference beam as small as possible. Since the master hologram will have the same wavelength light for both recording and illuminating, the difference between the recording and illuminating angles will not be great. It is therefore usually relatively easy to rotate the master hologram when illuminated to compensate for changes in the angle due to shrinkage. Therefore the angles associated with the master hologram are often dictated by the need to minimize difficulties in mounting the object. It is for this reason that the worksheet requires that you choose θO and θR_1 and then calculate θS_1 and θPD_1. You must also enter values for the reference distance to H_1, the illuminating distance to H_1, and the shrinkage factor for the master. The formulas on lines No. 18-26 are then used to determine the angles and distances for the master camera.

If it is necessary to design the master camera by first fixing and , the following substitutions can be made in the worksheet:

line 21 $\quad \phi_1 = \dfrac{\theta PD_1 - \theta S_1}{2}$

line 22 $\quad db_1 = \dfrac{\cos \phi_1 \; (\lambda L)}{SH\,(\sin \theta PD_1 + \sin \theta S_1)}$

line 24 $\quad \beta rec_1 = \sin^{-1}\left(\dfrac{\lambda L}{2\ db}\right)$

line 25 $\quad \theta R_1 = \beta rec_1 + \phi_1$

line 26 $\quad \theta S_1 = \beta rec_1 - \phi_1$

If problems are encountered when attempting to implement the designed cameras (i.e., the object distance or slit distance is too small to permit the introduction of the reference beam), the final

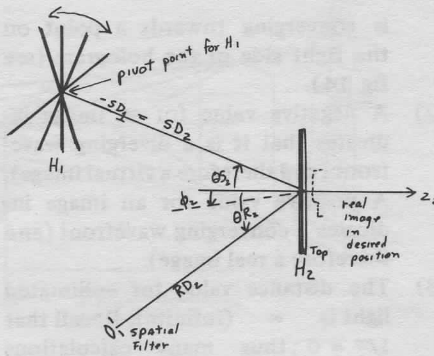


Fig. 16: Rotating H^1 around its pivot point to find the best location when illuminated in the transfer camera.

section of the worksheet, lines No. 27-32, can be used to modify the camera. To do so, first locate the line in the distance revision section containing the parameter that must be changed. Work down the sheet following the instructions. Each step presents the option of changing either the variable(s) on the left or right of the "OR". If the variable on the left is changed, no further calculations in this section are necessary. If the variable on the right is changed, then additional changes will be necessary. When all the changes have been made and are acceptable, the revised values are entered on the appropriate lines in the section of the worksheet that calculates distances (lines No. 14-20). Finally, the parameters on the lines between the last revised entry and line No. 21 must be recalculated. NOTE: IF λV IS CHANGED, LINES 1-5, and 7-14 WILL ALSO HAVE TO BE RECALCULATED.

It may be necessary to revise the variables several times before arriving at an acceptable camera design. This worksheet may also be used as an adjunct to McGrew's graphical procedure. (7) If McGrew's method is used as a first approximation, problems such as exceeding the table dimensions will have been resolved and refinement of the parameters can usually be generated in one pass through the worksheet (if the image characteristics are acceptable).

Specifying the Input Parameters

Display Parameters:

Lambda-Viewing, λV , is the wavelength in microns of the dominant viewing color. When the WLT is illuminated with white light and viewed from the ideal viewing position, the color of the central portion of the image is the dominant viewing color. A typical rainbow hologram will be designed so that the center of the image is green (.550 μ). If the hologram is designed to produce an image that will remain relatively monochromatic for a range of viewing distances, then λV is the wavelength of the monochromatic image.

Theta Illuminating, θI , is the angle of the illuminating source as measured with respect to the hologram's normal (See fig. 15 and sign convention section).

Theta Viewing, θV , can be thought of as the angle between the line of sight and the normal to the hologram at the center of the plate. The line of sight is defined as the ray entering the viewer's eye that appears to come from a specified point of interest of the image. This reference point is usually in the center of the image, but need not be. It should be the center of interest, however, since the hologram is designed as if the viewer will be looking predominantly in this direction (see fig. 15 and section on sign convention). θV is also the angle that the light diffracted by the hologram makes with the normal. Because diffraction is wavelength dependent, the viewing angle will vary with the wavelength of the reconstructing light. When designing the hologram, one must specify θV , the viewing angle for the dominant color. Note that the choice of viewing angle designates an ideal viewing height as well as an ideal viewing distance. When evaluating the color characteristics of the image, θV can be thought of as the angle between the normal and the ray traveling from the center of the hologram to the center of the slit image. When the eye is placed at this slit, the image will appear in the dominant color. The viewing angle for the other colors can be calculated. Theta viewing for the other colors will have a wavelength subscript such as θV_{550} for yellowish-green.

Viewing Distance, VD , is the distance between the hologram and the intended viewer. At this distance, the image will appear the least distorted. Note that this is the viewing distance for the *dominant viewing color*. Because the focal length of a hologram is wavelength dependent, the ideal viewing distance will vary with color. When designing the holocamera, VD for the dominant color will be specified, while the viewing distances for other wavelengths will be calculated. These variables will have subscripts indicating color such as VD_{633}, VD_{450}, etc. VD is usually a positive value, and is often equal to 0^o.

Illuminating Distance, ID, is the distance between the final hologram, H_2, and its intended light source. ID can be measured directly if the lighting display exists, or calculated if the display is to be created. If you plan to use the sun or another collimated source, then ID = ∞ . Because $1/\infty = 0$, certain formulas are significantly simplified (see sign convention section).

Transfer Camera Parameters:

Lambda-Laser, λL , is the wavelength of the laser used during the recording process, measured in microns. Some commonly used values include: ruby, .694 μ ; helium-neon, .633 μ ; and argon with .514 μ and .458 μ .

Shrinkage Factor, SH, is the percentage of the recorded fringe distance that remains after the emulsion shrinks due to processing. SH usually lies between .98 and .94. SH_2 is the shrinkage rate of the WLT, and SH_1 is the shrinkage rate of the master. Because master holograms are usually not bleached, and WLT holograms are often bleached, the shrinkage rates are not always the same.

Reference Distance (2), RD_2, is the distance between the pinhole of the reference beam's spatial filter and the center of the transfer hologram, H_2 (see section on sign conventions). Ideally the reference distance should equal the time-reverse of the illuminating beam, (RD_2 = -ID). When this is not possible, it is best to make both RD_2 and ID as long as possible.

Master Camera Parameters:

Shrinkage Factor, SH_1, is the shrinkage rate of the master.

Reference Distance (1), RD_1, is the distance between the pinhole of the reference beam's spatial filter and the center of the master hologram, H_1 (see comments under RD_2, above).

Theta-Reference (1), θR_1 , is the angle between the normal to H_1 and the ray traveling from the reference source to the center of the hologram (see fig. 14

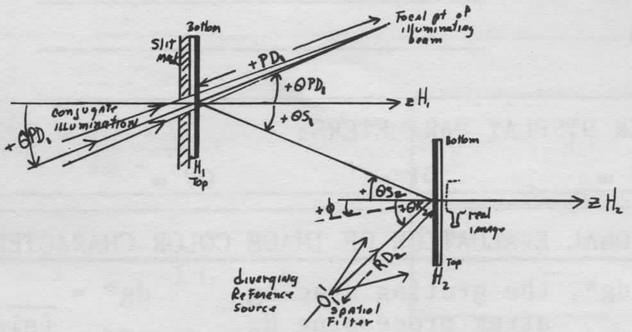


Fig. 17: In this diagram the master hologram, H^1, is illuminated by a converging illuminating beam – therefore PD is positive and is measured from H^1 to the point where the beam comes to a focus. The reference source for H^2 is a diverging beam, therefore RD is negative and is the distance between the pinhole of the reference spatial filter and H^2.

and section on sign conventions).

Theta-Object, θO , is the angle between the normal to H_1 and the ray traveling from the center of the object to the center of the hologram. θO is subject to the same sign convention as θV (see discussion under θV and the section on sign conventions).

Pinhole Distance (1), PD_1, is the distance from the pinhole of the spatial filter in the object beam to the center of the slit in the mask covering H_1. If the illuminating beam is diverging, then PD_1 is negative. If the beam is converging towards a point on the other side of H_1, then PD_1 is positive. Note that PD_1 is measured with respect to the z-axis of the master, not the z-axis of the WLT (see fig. 17 and section on sign conventions). If collimators are available, use them to record and illuminate H_1, to minimize the distortion in the final image. When collimators are used $PD_1 = \infty$.

Units to be Used:

All wavelengths should be entered in microns (10^{-6} meters). The fringe spacing calculated (dg, db) will also be in microns and the spatial frequency, ν , will be in lines per mm. All distances and focal lengths must be in the same unit. However, you may use either centimeters or inches as long as you are consistent. This is an option because the wavelength ratios (λV / λL , λL / λV) used in the distance formulas are unitless.

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WLT HOLO-CAMERA WORKSHEET

Project title ______________________

Date ______________________

ENTER DISPLAY PARAMETERS:

λV = ΘI = ΘV =   VD =   ID =

OPTIONAL EVALUATION OF IMAGE COLOR CHARACTERISTICS (See text for Procedure)

1.	dg^*,	the grating spacing after processing H_2	$dg^* = \dfrac{\lambda V}{(\sin \Theta I + \sin \Theta V)}$	dg^* =
2.	ΘV_u,	the diffraction angle of .67micron light	$\Theta V_u = \sin^{-1}\left(\dfrac{.67}{dg^*} - \sin \Theta I\right)$	ΘV_u =
3.	ΘV_ℓ,	the diffraction angle of .42micron light	$\Theta V_\ell = \sin^{-1}\left(\dfrac{.42}{dg^*} - \sin \Theta I\right)$	ΘV_ℓ =
4.	ΘV_i,	angles for calibrating the spectral window	ENTER:	
5.	λ_i,	wavelengths corresponding to ΘV_i	$\lambda_i = dg^* (\sin \Theta I + \sin \Theta V_i)$	
6.	α,	spectral axis angle	$\alpha = \tan^{-1}(\sin \Theta I)$	α =

ΘV_1 =	ΘV_2 =	ΘV_3 =	ΘV_4 =	ΘV_5 =
λ_1 =	λ_2 =	λ_3 =	λ_4 =	λ_5 =

ENTER TRANSFER CAMERA PARAMETERS:

λL = SH_2 = RD_2 =

TRANSFER CAMERA DESIGN

7.	dg_2,	grating spacing to be recorded in H_2, adjusted for shrinkage	$dg_2 = \dfrac{\lambda V}{SH_2(\sin \Theta I + \sin \Theta V)}$	dg_2 =
8.	ϕ_2,	angle the fringe plane makes with the normal	$\phi_2 = \dfrac{\Theta I - \Theta V}{2}$	ϕ_2 =
9.	db_2,	the Bragg spacing between fringe planes in H_2	$db_2 = dg_2 \cos \phi_2$	db_2 =
10.	ν_2,	spatial frequency to be recorded in H_2	$\nu_2 = 1000/db_2$	ν_2
11.	βrec_2,	Bragg recording angle	$\beta rec_2 = \sin^{-1}\left(\dfrac{\lambda L}{2\, db_2}\right)$	βrec_2=
12.	ΘR_2,	reference angle, H_2	$\Theta R_2 = \beta rec_2 + \phi_2$	ΘR_2 =
13.	ΘS_2,	slit angle measured from normal to H_2	$\Theta S_2 = \beta rec_2 - \phi_2$	ΘS_2 =

TRANSFER CAMERA continued

14.	fV_2,	focal length of H_2 when back-illuminated with viewing color	$fV_2 = \frac{ID \cdot VD}{ID - VD}$ *	$fV_2 =$	DISTANCE CALCULATIONS
15.	$fpri_2$,	primary focal length to be recorded in H_2	$fpri_2 = -\frac{\lambda V}{\lambda L} fV_2$	$fpri_2 =$	
16.	SD_2,	slit to H_2 distance as measured from H_2, will be negative value	$SD_2 = \frac{RD_2 \cdot fpri_2}{RD_2 + fpri_2}$	$SD_2 =$	
17.	SD_1,	H_1 to H_2 distance	$SD_1 = -SD_2$	$SD_1 =$	

ENTER MASTER CAMERA PARAMETERS:
$SH_1 =$ $RD_1 =$ $\theta R_1 =$ $\theta O =$ PD_1

MASTER CAMERA DESIGN

18.	$fconj_1$,	desired conjugate focal length to focus image reference pt at H_2 plane	$fconj_1 = \frac{PD_1 \cdot SD_1}{PD_1 - SD_1}$ *	$fconj_1 =$
19.	$fpri_1$,	primary focal length to be recorded in H_1	$fpri_1 = -fconj_1$	$fpri_1 =$
20.	OD_1,	object distance for H_1	$OD_1 = \frac{RD_1 \cdot fpri_1}{RD_1 + fpri_1}$	$OD_1 =$
21.	ϕ_1,	angle fringe planes make with normal to H_1	$\phi_1 = \frac{\theta R_1 - \theta O}{2}$	$\phi_1 =$
22.	db_1,	Bragg fringe spacing recorded in H_1	$db_1 = \cos\phi_1 \left(\frac{\lambda L}{\sin \theta R_1 + \sin \theta O} \right)$	$db_1 =$
23.	ν_1,	spatial frequency recorded in H_1	$\nu_1 = 1000/db_1$	$\nu_1 =$
24.	βill_1,	Bragg angle of illumination for H_1	$\beta ill_1 = \sin^{-1} \left(\frac{\lambda L}{2\, db_1\, SH_1} \right)$	$\beta ill_1 =$
25.	θPD_1,	illumination angle measured from normal to H_1	$\theta PD_1 = \beta ill_1 + \phi_1$	$\theta PD_1 =$
26.	θS_1,	slit angle measured from normal to H_1	$\beta ill_1 - \phi_1 = \theta S_1$	$\theta S_1 =$

** If ID or $PD_1 = \infty$ then $fV_2 = VD$ or $fconj_1 = SD_1$.*

DISTANCE REVISIONS

New Values

27. OD_1, Change as necessary, then alter either RD_1 or $fpri_1$: $OD_1 =$

28a. $RD_1 = \dfrac{fpri_1 \cdot OD_1}{fpri_1 - OD_1}$	OR	28b. $fpri_1 = \dfrac{RD_1 \cdot OD_1}{RD_1 - OD_1}$	=
		28c. $fconj_1 = -fpri_1$	=

If $fpri_1$ and $fconj_1$ have new values then you must alter either PD_1 or SD_1:

29a. $PD_1 = \dfrac{fconj_1 \cdot SD_1}{fconj_1 - SD_1}$	OR	29b. $SD_1 = \dfrac{fconj_1 \cdot PD_1}{fconj_1 + PD_1}$	=
		29c. $SD_2 = -SD_1$	=

If SD_1 and SD_2 have new values, then you must alter either RD_2 or $fpri_2$:

30a. $RD_2 = \dfrac{fpri_2 \cdot SD_2}{fpri_2 - SD_2}$	OR	30b. $fpri_2 = \dfrac{RD_2 \cdot SD_2}{RD_2 - SD_2}$	=

If $fpri_2$ has a new value, then alter either λV or fV_2:

31a. $\lambda V = -\dfrac{fpri_2 \cdot \lambda L}{fV_2}$ *see footnote	OR	31b. $fV_2 = -\dfrac{fpri_2 \cdot \lambda L}{\lambda V}$	=

If fV_2 has a new value then either ID or VD must change:

32a. $ID = \dfrac{fV_2 \cdot VD}{fV_2 - VD}$	OR	32b. $VD = \dfrac{fV_2 \cdot ID}{fV_2 + ID}$	=

* If λV is changed, the color characteristics of the holographic image will also change. Moreover all the diffraction parameters (lines #7-13) of the transfer camera will have to be recalculated.