THE MATHEMATICAL OPTICS OF WHITE-LIGHT TRANSMISSION HOLOGRAMS

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INTRODUCTION

Because white-light transmission holograms direct their images to very well-defined viewing locations, in order to make the best use of the illuminating light, they must be carefully designed to make sure that the diffracted light does indeed reach the intended viewer. This requires us to consider the hologram as a combination of prism and lens optical elements, so that it changes the angle of the light, and focuses it to the viewer's eye. But these are now diffraction prisms and lenses, holographically created, so that their properties vary with the wavelength, or color, of the light being used. Generally, we use one color of light, helium-neon red for example, to make holograms designed for the best viewing in green light (so that their performance is roughly balanced over the visible spectrum). In addition, the distances we must use on the holography table are much different (usually shorter) than those used in viewing. All of these changes can be compensated for by taking into account only two mathematical laws of diffraction and focusing, but these do require a certain amount of algebraic manipulation, with occasional use of trig functions and other features of today's pocket calculators. In the end, every holographer will settle on the design of a standard holo-camera that will fit on a particular table, and produce holograms for the viewing conditions decided upon. Once bolted down, that camera is unlikely to change (except for object illumination arrangements), but figuring it out in the first place, and adapting it to new uses, will require grappling with these mathematics.

In this brief paper, we can only introduce the necessary ideas, and work them through in simple terms. We will not discuss why light is diffracted and focused in the ways that it is, but simply present and discuss the mathematical formulas that describe by what angles it will be deviated, and at what distances it will be focused. In each case, we will reduce the formulas to actual prescriptions for practice, with diagrammed table setups. With the help of these notes, one should be able to backtrack, and see how the mechanics of holographic shooting relate to the mathematical physics of light waves. We acknowledge at the outset that it is impossible to digest all of this material on the first pass, especially for those for whom this is an introduction to rainbow holography. But these ideas have been used successfully by many practicing holographic artists, so we encourage you to persevere!

HOLOGRAPHIC OPTICS

A rainbow hologram can be thought of as a particular type of optical element that directs light from an illumination source, such as the overhead sun, toward the eye of an observer some distance away, perhaps at arm's length, as in Figure 1a. That redirection is modulated by image information, and that information changes for various side by side viewer

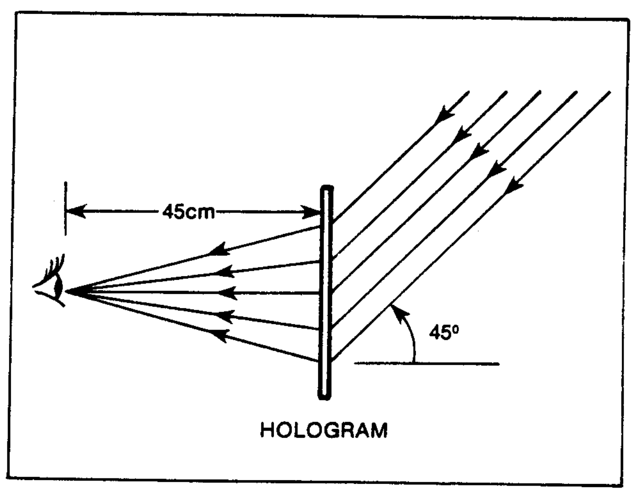

FIGURE 1a. HOLOGRAM EQUIVALENTS

positions in order to give a 3-D stereo pair to the two eyes, and to provide a "look around" effect. In what follows, we will simplify this to only an open frame visible at a single central viewer position, but the same type of analysis can be applied to other positions in the master hologram slit and viewer zone to calculate some of the more subtle image distortions. This analysis will deal mostly with viewer position and image color.

This greatly simplified hologram can therefore be considered as a combination of a prism, to deflect the sunlight toward the viewer's face, and a converging or positive lens to focus the sunlight right into his eye, as in Figure 1b. The problem then is to specify just the right prism power for the needed deflection angle, and the correct lens power for the focusing distance, and then find ways to generate those optical powers holographically.

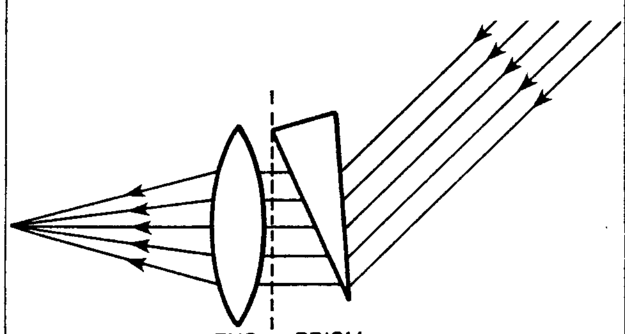

FIGURE 1b. HOLOGRAM EQUIVALENTS

However, as Figure 1c shows, these are not glass optical elements, but their diffraction or holographic replacements. Because holographic optical properties depend on the wavelength of light used, we have to specify what color we want them to operate at, and then see how the hologram works in other colors.

Thus we will start with two mathematical formulas, which we will not derive here, that describe the wavelength dependent properties of holographic prisms (that is, diffraction gratings) and holographic lenses (which are approximate Fresnel zone plates). After figuring out how the hologram will behave, we go on to figure out how to create such a hologram with laser light of an entirely different color, and with a table that is much shorter than the sun is far away. This, in turn, will specify what properties we need from the first or "master" hologram, so that its exposure geometry can be determined. Note that we work backwards from the end result to the beginning object. In actual practice, holographers have to crank these formulas backward and forward many times before they feel confident in making any theoretical predictions at all. But the first phase will usually be backwards, starting with what one hopes the final result will be like, and trying to find a set-up that will fit on the table!

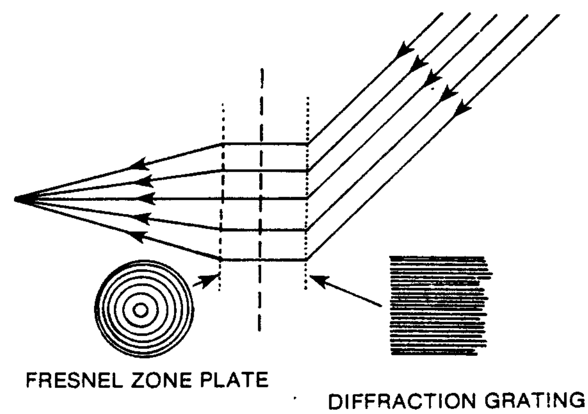

FIGURE 1c. HOLOGRAM EQUIVALENTS

HOLOGRAPHIC PRISMS — Diffraction Angles

When a beam of light passes through any regularly spaced structure, part of the light is deviated away from the beam in two oppositely directed parts by a process called diffraction. When the repetitive spacing is so small as to approach the wavelength of light, the angles of deviation become substantial, and the structure is often referred to as a "diffraction grating". Ordinarily, such gratings are made by mechanically scratching parallel lines into a glass plate, perhaps one scratch every micron (one thousand lines per millimeter, twenty five thousand per inch, or fifty in the width of a human hair!). Those equally-spaced parallel lines can be even better engraved by holography, as we shall see. The basic law for the amount of deviation by a diffraction grating is called the "diffraction equation", and it will be the first key in designing a rainbow hologram.

Referring to Figure 2a, let θI be the angle of illumination, and θV be the angle of view, both measured from the perpendicular to a grating of spacing d (the grating need not be vertical). If the illuminating light has a wavelength lambda (λ), the angles are related by

$$\sin(\theta V) + \sin(\theta I) = \lambda/d.$$

There is also a beam deviated downward, which we will ignore here (it is often much weaker).

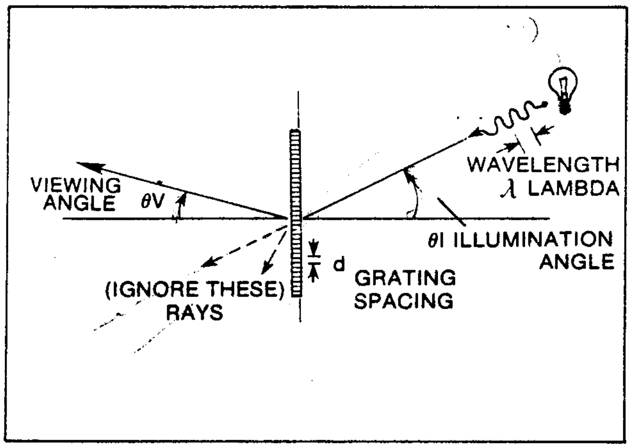

FIGURE 2a. GENERAL CASE DIFFRACTION

Thus, if we want to take a λ = 0.55 micron ("green") beam coming from 45 degrees above, and deviate it to be travelling horizontally (see Figure 2b), the required grating spacing, d, is determined by

$$\sin 0 + \sin 45 = .55/d,$$

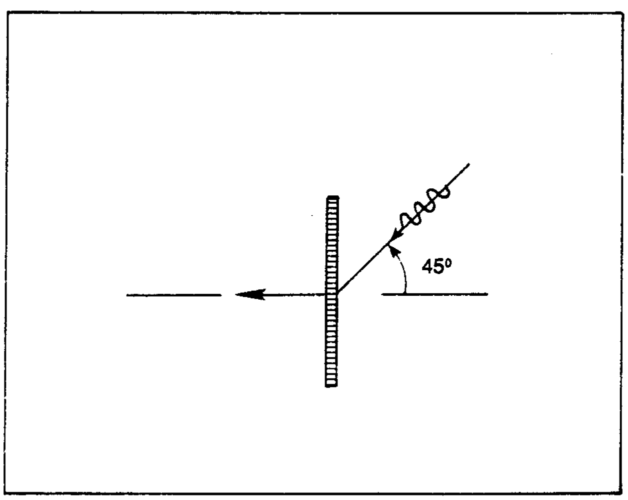

FIGURE 2b. EXAMPLE θV = O, θ = 45°, λ = 0.55 μm d = ?

so that the grating spacing is

$$d = 0.78 \text{ micron},$$

which can also ge described by its "spatial frequency"

$$f = 1000/d = 1300 \text{ cycles/mm}.$$

Now we can ask what happens if the illumination beam is white light, that is, if it has "red" (λ = .63 micron) light and "blue" (λ = .47 micron) light as well as "green" light. Where do those other colors go? Now we insert "d", and look for the red-light angle, θV_{red};

$$\sin(\theta V_{red}) + \sin 45 = .63/.78,$$

and find that

$$\theta V_{red} = 5.8 \text{ degrees}.$$

Similarly for blue light,

$$\sin(\theta V_{blue}) + \sin 45 = .47/.78,$$

so that

$$\theta V_{blue} = -6.0 \text{ degrees}.$$

Now we can sketch the first part of the answer to "where do the colors go?" Figure 3a shows the calculated angles plotted out, showing that the rainbow viewing zone expands vertically as we move away from the hologram, subtending a constant fan angle. Figures 3b and 3c show other examples of diffraction by gratings, showing that a coarser grating gives a narrower diffraction fan, while a finer grating gives an only slightly wider diffraction fan.

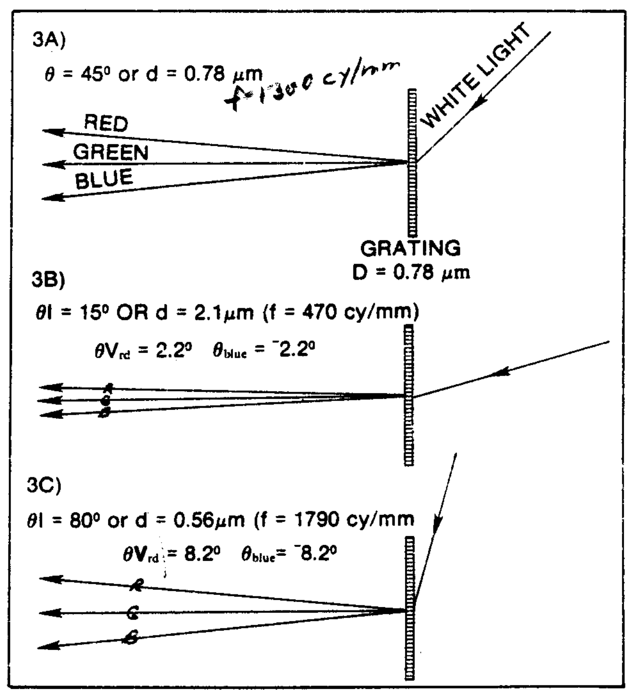

FIGURE 3. DIFFRACTION OF "WHITE LIGHT"

REFRACTION FOCUSING — Glass Lenses

Figure 4 shows the focusing of rays by a normal positive lens. The object is taken to be a point source of light, so that if the pupil of the eye is placed in the focused real image, the entire lens lights up. The object and image distances are related by "focusing laws", which can be formulated in various ways. Perhaps the best known is this one:

(1/VD) + (1/ID) = (1/F.L.),

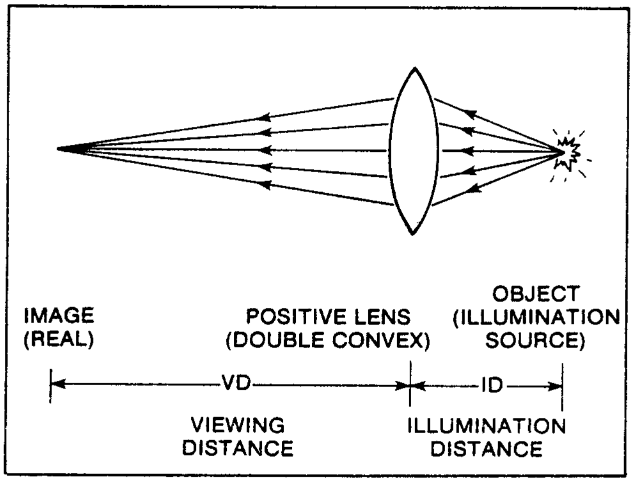

FIGURE 4. GENERAL CASE: REFRACTIVE FOCUSING BY GLASS LENSES

The distance to the source, or illumination point, is ID (for Illumination Distance), while the distance to the real image, or viewing position is VD (for Viewing Distance), and F.L. is the focal length of the lens. The focal length is a fixed property of a lens, determined by the curvatures of its surfaces and the type of glass it is made from. It can be measured as the distance from the lens to an image of the sun, as when the lens is used as a burning glass. The ID is always positive, but if it is less than the Focal Length of the lens, VD will come out negative, which means that a so-called "virtual" image is formed, that can be seen behind the lens. For larger ID, the VD is positive, and a so-called "real" image is formed in front of the lens, where you can focus it on a white card, or place your eye to see the whole lens light up at once.

DIFFRACTION FOCUSING — Holographic Lenses

If a diffraction grating has a spacing that varies slowly, then rays from different parts of the grating will emerge non-parallel, and can overlap at a distance from the grating. If th variation is just right, whether mechanicall scribed or holographically produced, such a element can behave just like a lens, and it focusing is governed by the same formula tha governs a lens, except that the focal length o "power" of the lens changes with th wavelength of light being focused. Actually the situation can be much more complex if w go on to consider the defects in lenses, glas or holographic, and even here we have t make a choice as to whether we will discus the focusing in the plane of the paper, o perpendicular to that plane. The first focu determines exactly where the focuse rainbow spectrum will appear, while th second determines the distance at which th image will appear to be the least distorted. W think that the second calculation is mor important, and its equations are simpler too so that is the focus we will be discussing here. The other focus may be located by dividing all of the distances by the square of the cosines of their associated angles (see S.A. Benton, "Aberrations of Holographic Wavefronts", Journal of the Optical Society of America, 1824A, December 1982), and is generally somewhat closer to the hologram.

Figure 5 shows a conceptual holographic lens focusing experiment. Again, the distance to the illumination point source is ID (Illumination Distance). The holo-lens is a positive, or magnifying-type lens, and forms a real image of the source at some distance. This is called the Viewing Distance, VD, because an eye placed here will receive light diffracted by the entire hologram, so that the frame will appear to glow brightly.

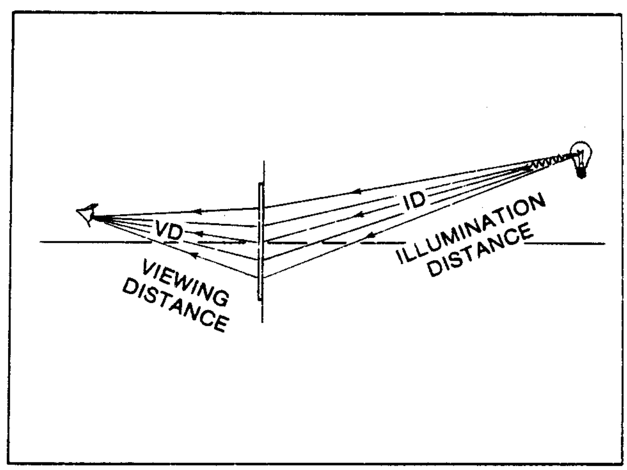

FIGURE 5. DIFFRACTIVE FOCUSING BY HOLOGRAPHIC LENSES

The difference with a holographic lens is that the object, lens and image are not usually on a straight line (due to the effect of the holo-prism), and that the focal length of the lens depends on the color of the light, being shorter for red light than blue light. Thus the viewing distances (and angles) will be different for each color. Here we have deleted all the angles, because we are concentrating on the lens properties of the hologram, and the distance relationships that they imply. The focusing relations for a holo-lens can be written as:

$$(1/VD) + (1/ID) = (\lambda/LF).$$

Here LF signifies the product of the wavelength that the lens was made with, and the focal length of the holo-lens when used at that wavelength (call it "lambda focus"). In terms of the glass-lens analogy, LF is the fixed intrinsic property of the diffraction lens, and is established during its exposure. λ is the present wavelength of interest for viewing.

For example, Figure 6 shows the rainbow hologram represented as a holo-lens, focusing light from an infinitely distant source, the sun, to a real image 45 cm away, at the viewer's eye, in green light. Fitting these data into the focusing law gives:

$$(1/45) + (0 = 1/\infty) = (.55/LF),$$

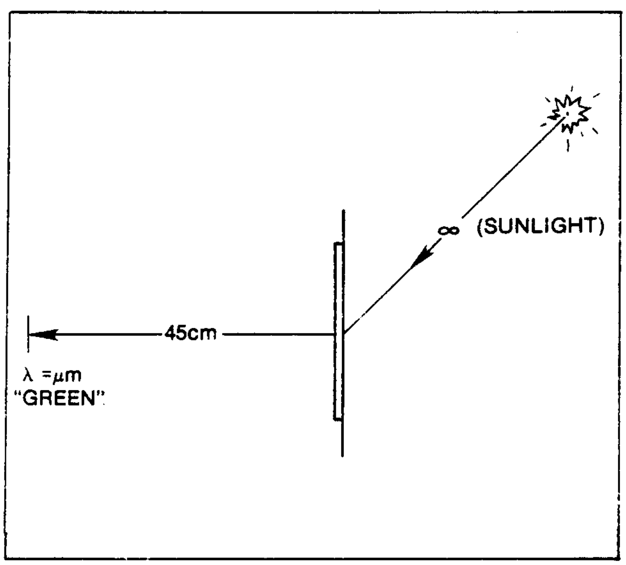

FIGURE 6. EXAMPLE

so that solving for LF gives

$$LF = 25 \text{ (microns x centimeters)}.$$

Where is the "red" light? Let λ be 0.63 microns, so that

$$(1/VD_{red}) + 0 = (.63/25),$$
$$VD_{red} = 25/.63 = 40\text{cm}.$$

And where is the "blue" light? Let λ be 0.47 microns, so that

$$(1/VD_{blue}) + 0 = (.47/25), \text{ and}$$
$$VD_{blue} = 25/.47 = 53 \text{ cm}.$$

The diffraction grating equation and diffraction focusing equations have given us the angles and distances to the viewing locations for red and blue light for a rainbow hologram designed for optimum viewing in green light. We can plot these out for these wavelengths, and all those in between, as in Figure 7. Here we see a spectrum dispersed in space, as one normally sees for a rainbow hologram. Note that it is indeed tall enough, about 10 cm, to present a practical viewing window to our eyes. But also note that the spectrum is inclined, sharply tipped toward the light source at an angle that we will call "alpha", or the "achromatic angle", which is used in the design of multi-color holograms. Although we can roughly measure alpha from the plot, it can also be calculated in a general way as:

$$\alpha = \text{arc-tangent} (\sin \theta I).$$

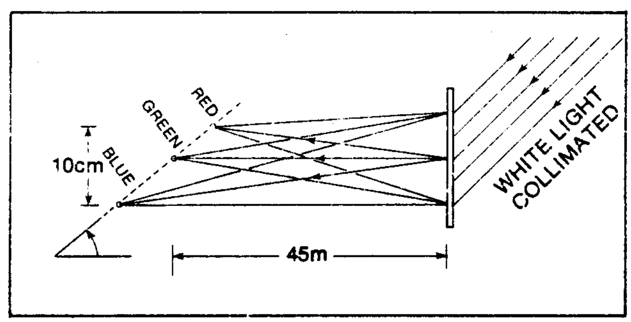

FIGURE 7. SPECTRAL PROCESSING

Sample values of α for different illumination angles are shown in Table 1.

TABLE 1.

θI	30°	45°	60°
α	26½°	35°	41°

While simpler versions of the optics of holographics lenses exist, such as the "hinge point" graphical analysis of the off-axis lens model presented by Stephen McCrew at this Symposium, they mis-predict the achromatic angle (as $\alpha = \theta I$, in this case), and some other focal behaviors, and should be used only as rough first approximations. The "prism/lens" model described here is exact for each focal plane, as already described, which is required for high quality multi-color and achromatic imaging.

These example calculations have been for a typical hand-held hologram, but can easily be extended to a typical display hologram, where the illumination distance is rather shorter, say two meters, and the intended green-light viewing distance is rather longer, perhaps also two meters. We now move on to consider how the same two formulae may be applied to the problem of making the hand-held hologram on the holography table.

INTERFERENCE — Making a Holo-Prism

When two coherent plane waves overlap at an angle, they form interference fringes throughout their intersecting spaces, and if a white card or a photographic plate is inserted within that space to observe or record those fringes, they will appear with a spacing that depends on the angles of the beams to the perpendicular to the plate. This relationship is also described (by no small coincidence!) by the diffraction grating equation we saw before:

$$\sin(\theta O) + \sin(\theta R) = \lambda/d,$$

where θO and θR are the angles of the descending object beam and the ascending reference beam with respect to the perpendicular to the plate, as shown in Figure 8. Here "d" is again the spacing between the grating lines produced. This relation also holds true if the beams are not quite plane waves, but coming from point sources that are much further away than d, or the width of the area we are interested in exploring at the moment, that is, greater than a few centimeters away.

Our goal is to produce a grating with a spacing of d = 0.78 micron, which will give a 45 degree deflection of "green" light, but made presumably with He-Ne laser light of wavelength 0.63 microns. We see immediately that there are many combinations of reference and object beam angles that will produc same d, if they are varied together, so does one pick the best combination? This tricky point, because what we have said s really assumes that we are talking about thin holograms, whereas most ac holograms are three or more microns th which introduces some complications. We going to continue to ignore th complications, except to say that its a g idea to keep either the object or refere beam near the intended angle of use (t can't both match because of the wavelen change). Even this isn't adequate if emulsion shrinks or swells significan between exposure and viewing, but t analysis must await a further Symposium.

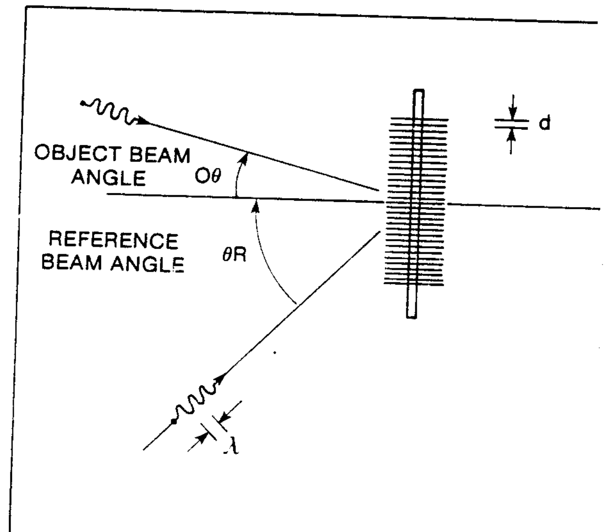

FIGURE 8. INTERFERENCE: MAKING HOLO-PRISM

So let's try the first option, and set the objec beam where the nominal viewing beam will be perpendicular to the plate, so $\theta O = 0.0$, and

$$\sin 0 + \sin \theta R = .63/(d=.78),$$

so that

$$\theta R = 54 \text{ degrees}.$$

Or, try the second option, and set the object beam the same direction that the illumination beam will be coming from, so that θR is 45 degrees, and then

$$\sin(\theta O) + (\sin 45 = .707) = 63/.78,$$

and

$$\theta O = 5.8 \text{ degrees}.$$

The conceptual difference between fixing the object or reference beam angle gives rise to two distinct approaches to multi-color holography. But this doesn't make much practical difference, as the angle between the object and reference beams varies only between 54 and 51 degrees, and it becomes mainly a question of where you "point the plate" between them (if the shrinkage is 7%, which is typical for PAA-developed/bromine water-bleached plates, the angle between the beams ought to be 52.2 degrees, with the plate turned in 3.6 degrees from the object beam). A bigger problem is getting the separation of the master and transfer plates right, in order to make the correct holo-lens.

INTERFERENCE — Making a Holo-Lens to Order

Here again, we retread the by-now familiar lens focusing law to become the "lens exposing law", but the terms have to be redefined somewhat, as in Figure 9. Here OD (Object Distance) is the distance to the object beam source (it can be the center of a slit in the master hologram), and RD (Reference Distance) is the distance to the reference beam source, which we assume is larger than OD. The formula is then:

$$(1/OD) - (1/RD) = \lambda/LF.$$

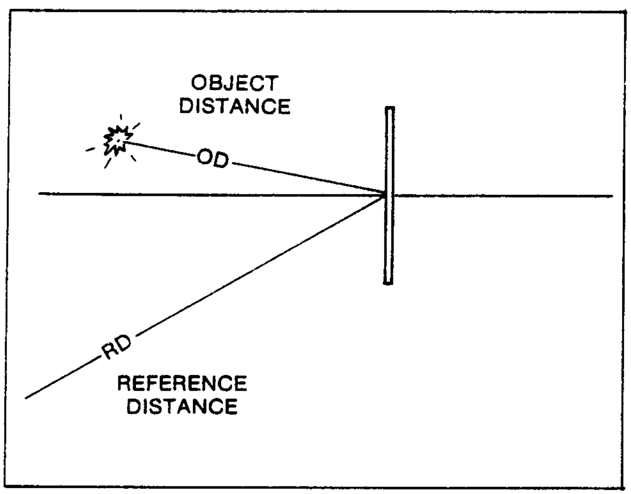

FIGURE 9. INTERFERENCE: MAKING A HOLO-LENS

Note well the "minus" sign between the left-side terms! We want to produce an LF of 25, but again we have an infinite choice of OD and RD combinations to choose from. Where do we start? In order to minimize distortion of the three-dimensional image, we should make the reference beam as nearly "conjugate" to the intended illumination beam as possible. If the reference beam is diverging from a point source, the reference beam should ideally be converging, focused at the location of that source.

Barring your possession of large diameter converging optics of high quality, the best bet is to use the very longest reference beam throw that can be managed on the table. Bounce it off mirrors if you have to (and if they are clean and stable enough), but get it out there. The distance should be at least ten times the largest plate dimension, unless you have already anticipated a lot of distortion and "image-swinging". We will use 150 cm for these examples, which ought to be easy, and is adequate for small holograms.

So, assuming a fixed reference distance of 150 cm, we have:

$$(1/OD) - (1/150) = .63/25,$$

so that

$$OD = 31.4 \text{ cm}.$$

This is the necessary separation between the master and transfer plates, H1 and H2, to give an image that straddles the hologram plane, and is the least distorted when viewed at arm's length. Note that the separation is markedly less than the 45 cm green viewing distance, partly because we don't have the ideal conjugate reference beam, and partly because we are shooting in the red light. This difference will cause a mild amount of forced perspective in the image, which can be quite dramatic and pleasing. Figure 10 graphically summarizes the various transfer geometries, differing slightly in angles, that give the desired viewing result. Review this briefly, before going on to consider how to produce a master hologram, H1, that projects the desired image at 31 cm, with and without collimators.

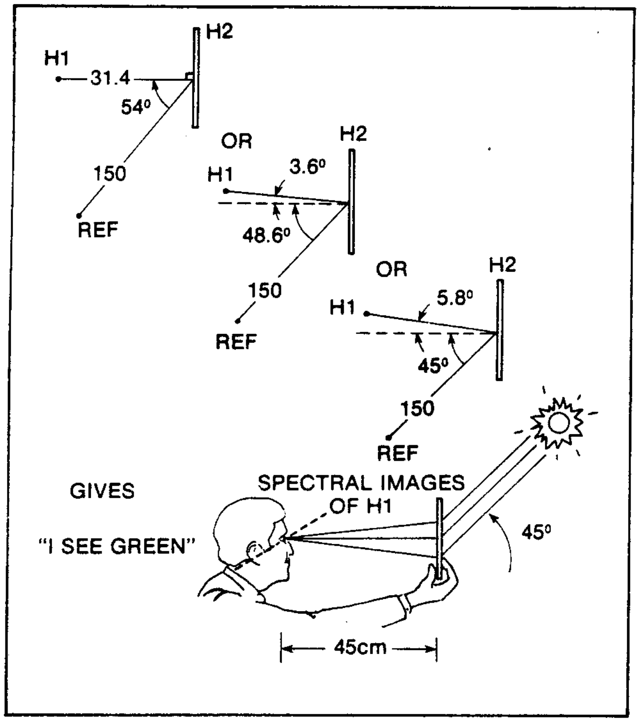

FIGURE 10. RAINBOW HOLOGRAMS

HOW TO PRODUCE AN H1 THAT PROJECTS AN IMAGE AT 31 CM

A. Without Collimators (150 cm beam throws)

The task is now to expose the master plate to an object at a distance such that its real image projection comes out exactly 31 cm away, and so will be in the plane of the rainbow hologram during exposure and viewing. This distance will serve as the reference distance for planning the object layout for exposure. The simplified reference object will be a point, so that we can also consider H1 to be a simple holographic lens-prism combination. The calculations are complicated only by the fact that no collimators are available, so that imperfect conjugates must be used for the reference and illumination (projection) beams. Again, we make these as nearly collimated as possible by using sources as far from the hologram as possible, 150 cm in these examples.

Figure 11 shows how the master plate, H1, can be considered as forming a real image of the projection beam point source, PD (for projection beam distance) away, at the center of the rainbow transfer, H2, at distance S (for separation) away, after deflection through an angle of θR (so that the projection and H2 reference beams are parallel for convenience). Because the same laser will be used for exposure and projection, the same angle can be used during exposure, saving us a diffraction prism calculation. The required diffraction lens focal length, LF, can be determined by

$$(1/PD) + (1/S) = \lambda/LF,$$

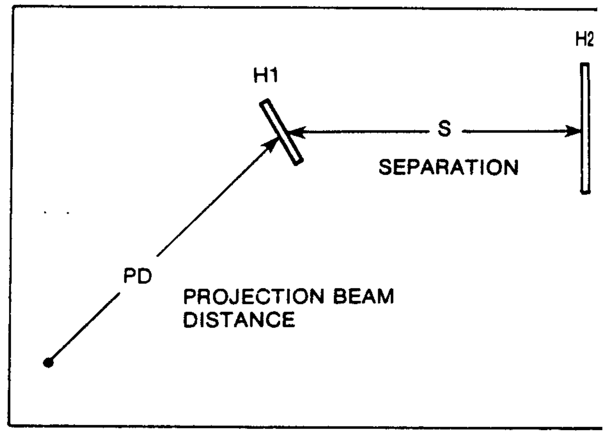

FIGURE 11. GENERAL CASE: PROJECTION

(this holds true even though only a narrow strip of H1 is actually used). Inserting the distances we have decided upon gives

$$(1/150) + (1/31) = .63/LF,$$

so that

$$LF = 16$$

Producing such a holo-lens with a diverging reference beam is illustrated in Figure 12. Again we use the interference formula

$$(1/OD) - (1/RD) = \lambda/LF.$$

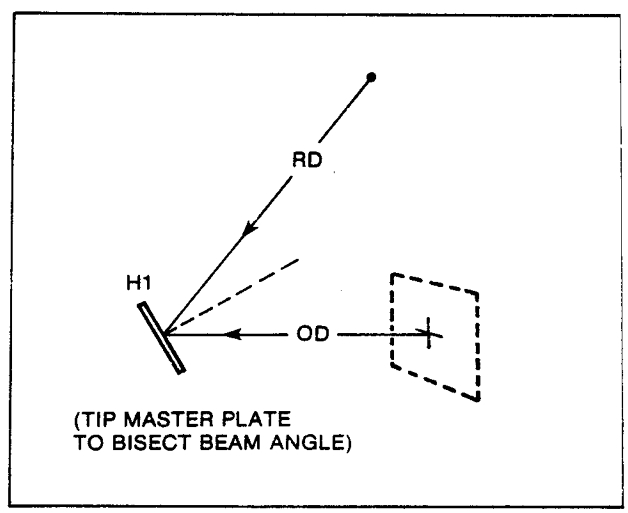

FIGURE 12. GENERAL CASE: EXPOSURE

Substituting the reference beam distance, 150 cm, and the required LF, 16, gives

(1/OD) = (.63/16) + (1/150),

OD = 22 cm.

Thus, objects that are intended to be in the rainbow hologram plane should be set up for 22 cm from the master plate during exposure. For purposes of previsualization and composition, we find the concept of a "virtual frame" very useful. This is a wire frame (often imaginary) placed at the OD distance, and sized so that after being projected at the S distance, it will be just the size of the intended hologram. In this case, after being recorded at 22 cm and projected at 31 cm, its image will be magnified by M = 31/22, so that the frame, and all the objects, must be made up at about two-thirds their intended size. It turns out that their magnification in depth will be the square of their side-to-side magnification upon projection, so they must be compressed by a factor of two. Thus a sphere will seem drawn out to a football-like shape (long direction out of the hologram), unless it is suitably pre-distorted! Even so, there will be some visual "swinging" of the image, and "rolling" in and out of the hologram plane as the viewer moves from side to side, because of other effects of the reference and projection beam mismatch. The worst of these can be avoided if collimators can be used for the recording and projection of H1, so if you have only one *collimator*, use it here and don't worry about the H2 reference beam.

B. Life With Collimators (RD = -PD = infinity)

Now exposures become much easier to calculate, and the images much less distorted. The objectes can be made the same size as intended, and set up with the reference plane at 31 cm from H1 during exposure. Because the collimated beams are self-conjugate, they will be projected at that desired distance without magnification or distortion, as illustrated in Figures 13a and 13b.

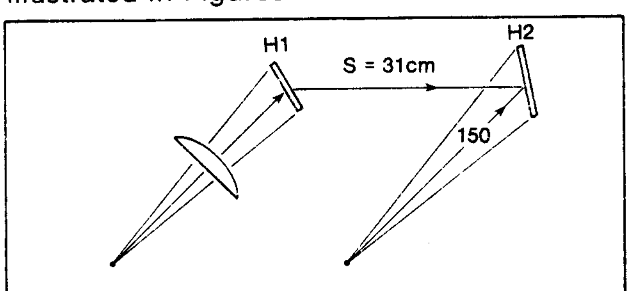

FIGURE 13a. PROJECTION

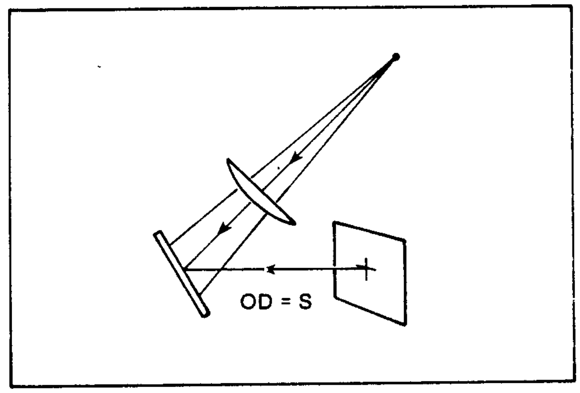

FIGURE 13b. EXPOSURE

C. Types of Collimators

Collimators are lenses, mirrors, or other optical elements that produce an image of a point source of light (which is placed at the "focal point") that appears to be an infinite distance away, like a star, so that the rays of light are parallel (or "collimated"). Because a collimator is such an important tool for the rainbow holographer, it is appropriate to say a few words about types to consider acquiring.

Glass: a single plano-convex lens will do very well, especially if the focal length is at least four times the lens diameter (which must be greater than the intended slit width). The flat side of the lens should face the pinhole. Anti-reflection coatings on both lens surfaces are recommended, to avoid a small but troublesome secondary reflection, but not absolutely necessary. Be sure to check the glass for bubbles and internal streaks ("striae") by scrutinizing the collimated beam several meters downstream.

Plastic: these lenses are lighter than glass, and much cheaper on a custom basis, but they are very fragile, and hard to put a good optical surface on. Be sure to check it for birefringence and orange peel effects in the collimated beam (most vendors can correct these for a price, if you specify that the lens is for a laser application). Anti-reflection coatings are also less satisfactory and more fragile when on plastic. Some holographers have been very happy with these, but I would keep looking for a bargain in glass.

Reflective: an off-axis section of a parabolic mirror can provide perfect collimation that is free from subsidiary reflections, and variations with wavelength. These are fairly rare, though sometimes can be found in obsolete optical instruments. Ordinary spherical mirrors (front-surfaced) have significant aberrations when used off-axis, but may be useful.

Holographic: this is probably a holographer's best bet for a first collimator, and is one of the most challenging types of holograms to make. You should be able to figure out what to do based on these notes. See Figure 14 for the relevant algebra. The resulting wavefronts will be noticeably aberrated, and may produce a slightly curved slit, but such a collimator is worth a try, especially if only a strip is needed.

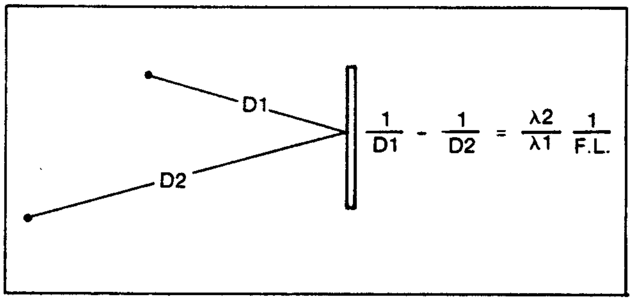

FIGURE 14. HOLO-COLLIMATOR EXPOSURE

CONCLUSION

Figure 15 summarizes the systems for producing hand-held rainbow holograms that we have worked out here. You should go on to work out similar systems for other hologram types, such as those for permanent display.

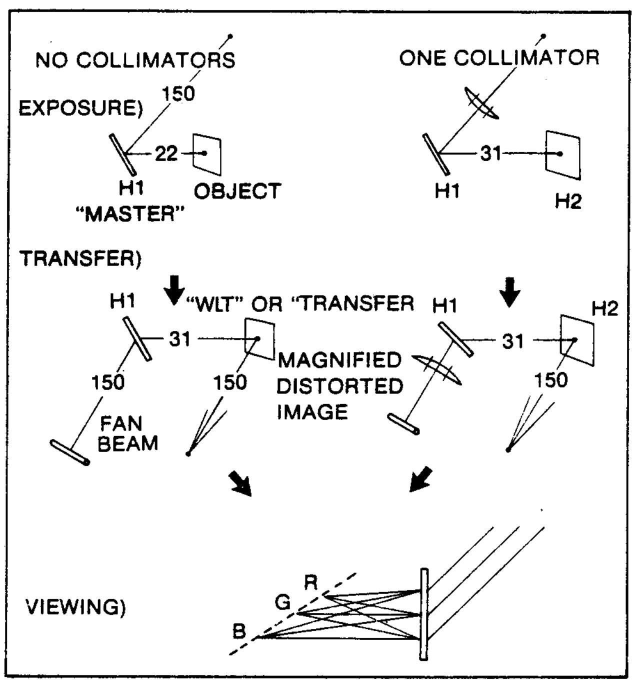

FIGURE 15. TWO RAINBOW SYSTEMS

Multi-Color and Achromatic Holograms

Space limits us here to a discussion of the detailed optics of "rainbow" type holograms but the same concepts can be extended to the design of multi-color and neutral-toned or "achromatic" holograms, as may be described in subsequent papers. The central concept is to produce superimposed images of master slits in several different wavelengths of light so that the images they produce will appear in register in several different colors. This can be done by either using a single master slit location and three (or more) reference beam locations for sequential exposures, or a single reference beam location and several slit locations for a single exposure. All the angles and distances can be worked out from the formulas provided here. In cases where the same image is to be projected by all the slits, to produce an achromatic image, a beam-splitting "diffractor plate" can be used to produce an array of virtual images of a single slit, aligned along the "achromatic angle", which provide the multiple object beams for exposure.

The amount of calculation and set-up precision that white-light transmission holography requires may be daunting to holographers accustomed to full-aperture transfers, perhaps in a sand-table environment. But the same considerations come into play in any advanced holographic technique, and with practice, their accommodation becomes a routine part of a holographer's practice. In any case, holographers soon learn their way through the forest of numbers and formulae, to develop a "feel" for workable set-ups that allows them to concentrate again on using the image space to its fullest advantage. We hope that this preliminary tour of the territory will help hasten that progress.